ABSTRACT

Heating and cooling load estimation is based on the heat balance method. Although this method has strong physical basis, it yields an acausal system when the heat capacity of the indoor space is non-negligible. New emergent problems in low energy buildings require time steps significantly smaller than 1 hour, which is usually used by simulation software. Therefore, when the time step tends to zero, the calculated load tends to infinity when the indoor air temperature has an incremental variation. This paper demonstrates that the heat balance method results in an acausal system and analyses the variation of load estimation with the size of the simulation time step.

INTRODUCTION

Load estimation is an essential task for building design, especially for HVAC sizing. It is also used to check the conformity with the thermal regulations. The basis for load estimation is the heat balance. It is recommended by ASHRAE (Bellenger et al. 2001) and used by the CEN standards (e.g. CEN 2003, CEN 2007a, CEN 2007b) and simulation software (TRNSYS 2007, EnergyPlus 2012, Clarke 2001, Skeiker 2010). The method is based on the heat balance of the air-volume of a thermal zone (Wang and Chen 2000, Bellenger et al. 2001, Hernandez et al. 2003, Xu and Wang 2008, Li et al. 2009):

$$ m_c \frac{d\theta}{dt} = q_c + q_v + Q_{HVAC} $$

(1)

where:

- $\theta$ is the air volume temperature considered homogeneous [°C or K];
- $m_c \frac{d\theta}{dt}$ is the heat accumulated in the air volume of mass $m_a$ and specific heat $c_a$;
- $q_c = \sum S_i h_i (\theta_i - \theta_a)$ is the convective heat flow from surface $S_i$ having the surface temperature $\theta_i$ and the inside heat convection coefficient $h_i$ [W];
- $q_v = m_v c_a (T_o - \theta_a)$ is the heat flow due to mass flow $m_v$ of outdoor air at temperature $T_o$ introduced into the space by infiltration and ventilation [W];
- $Q_{HVAC}$ is the convective heat flow gained from the internal loads [W];
- $Q_{HVAC}$ is the heat flow from the HVAC system [W].

The load, $Q_{HVAC}$, is determined from Equation (1). If the heat capacity of the room air is not neglected, then

$$ Q_{HVAC} = m_c \frac{d\theta}{dt} - q_c - q_v - Q_g $$

(2)

If the heat capacity of the air is neglected, then

$$ Q_{HVAC} = -q_c - q_v - Q_g $$

(3)

The room air temperature, $\theta_a(t)$, is considered known in Equations (2) and (3) because it is the desired temperature of the indoor air. Finding indoor air temperature $\theta_a$ from Equation (1) is a simulation (or direct) problem. It models a physical system and since all physical systems are causal (Romagnoli and Palazoglu 2012), the model is causal (i.e. its output depends only on past inputs). Finding the load $Q_{HVAC}$ from Equation (2) or (3) is a control (or inverse) problem. If the model for heat transfer is linear (or linearized), then equations (2) or (3) result in an improper transfer function which is an acausal model (i.e. its output depends also on future inputs).

The practical consequence is that in the case of non-negligible thermal capacity of the indoor air, when the indoor temperature varies, the value of the calculated thermal load will depend on the value of the time step used in simulation. The physical...
explanation is that the power needed to change the indoor temperature in exactly a time-step depends on the size of the time-step. Therefore, the size the HVAC system, which is chosen in function of the calculated load, will depend on the time-step used in simulation. Practitioners noticed this problem and introduced load limitations in the estimation of the load (e.g. CEN 2007b).

An epitome example will show how the transfer function can be obtained from the linear thermal model of a building, how this model is causal for the calculation of the indoor temperature and acausal for the thermal load, and, finally, the consequences on the results of load estimation.

Differential-Algebraic Equations

We will consider the model shown in Figure 1 with the values of the parameters given in Table 1. This network has the main characteristics of a thermal model. More elements may be added but the procedure and the results presented hereafter remain the same. The model of thermal zone is described by a set of differential algebraic equations that can be obtained from the way in which the components are connected in the circuit and the values of the circuit elements (Strang 1986, Ghiaus 2013). The thermal network from Figure 1 is described by the matrices and the vectors given in Figure 2. The temperature drops over the resistances are:

$$\mathbf{e} = -\mathbf{A}\mathbf{e} + \mathbf{b}$$  \hspace{1cm} (4) 

The heat flow rates are related to these temperature drops by the thermal conductivities:

$$\mathbf{q} = \mathbf{Ge}$$  \hspace{1cm} (5)
The heat balance for the indoor temperature, expressed by Equation (1), is the 3\textsuperscript{rd} row in this set of equations. The heat balance for each node gives:

\[
\mathbf{C}\mathbf{\dot{\theta}} = \mathbf{A}^T \mathbf{q} + \mathbf{f}
\]  

(6)

By eliminating \(\mathbf{e}\) and \(\mathbf{q}\) from the set of equations (4) - (6) we obtain a system of differential – algebraic equations:

\[
\mathbf{C}\mathbf{\dot{\theta}} = -\mathbf{A}^T\mathbf{G}\mathbf{\theta} + \mathbf{A}^T \mathbf{G}\mathbf{b} + \mathbf{f}
\]

(7)

that can be written as:

\[
\mathbf{C}\mathbf{\dot{\theta}} = \mathbf{K}\mathbf{\theta} + \mathbf{K}_b\mathbf{b} + \mathbf{f}
\]

(8)

where \(\mathbf{K} = -\mathbf{A}^T\mathbf{G}\) and \(\mathbf{K}_b = \mathbf{A}^T\mathbf{G}\).

**STATE-SPACE REPRESENTATION**

A more suitable form for system analysis is the state-space representation. In order to obtain it, the temperatures corresponding to nodes without thermal capacity need to be eliminated from the set of Equations (8). Partitioning the set of Equations (8),

\[
\begin{bmatrix}
0 & 0 & \mathbf{\dot{\theta}}_o \\
0 & \mathbf{C}_c & \mathbf{\dot{\theta}}_c
\end{bmatrix}
= 
\begin{bmatrix}
\mathbf{K}_{11} & \mathbf{K}_{12} & \mathbf{\theta}_o \\
\mathbf{K}_{21} & \mathbf{K}_{22} & \mathbf{\theta}_c
\end{bmatrix}
\begin{bmatrix}
\mathbf{\theta}_o \\
\mathbf{\theta}_c
\end{bmatrix}
+ 
\begin{bmatrix}
\mathbf{K}_{31} \\
\mathbf{K}_{32}
\end{bmatrix}
\begin{bmatrix}
\mathbf{b} \\
\mathbf{f}_c
\end{bmatrix}
\]

(9)

we obtain the state-space representation:

\[
\mathbf{\dot{\theta}}_o = \mathbf{A}_s\mathbf{\theta}_o + \mathbf{B}_s\mathbf{u}
\]

(10)

\[
\mathbf{\dot{\theta}}_c = \mathbf{C}_s\mathbf{\theta}_c + \mathbf{D}_s\mathbf{u}
\]

where

\[
\mathbf{\theta}_o\text{ and }\mathbf{f}_o\text{ correspond to the nodes without thermal capacity};
\]

\[
\mathbf{\theta}_c\text{ and }\mathbf{f}_c\text{ correspond to the nodes with thermal capacity};
\]

\(\mathbf{C}_c\) is the bloc of the partitioned matrix \(\mathbf{C}\) for which the diagonal elements are non-zero;

\(\mathbf{K}_{11}, \mathbf{K}_{12}, \mathbf{K}_{21}, \mathbf{K}_{22}\) are blocs of the partitioned matrix \(\mathbf{K}\) obtained according to the partitioning of the matrix \(\mathbf{C}\);

\(\mathbf{K}_{b1}, \mathbf{K}_{b2}\) are blocs of the partitioned matrix \(\mathbf{K}_b\) obtained according to the partitioning of the matrix \(\mathbf{C}\);

\(\mathbf{I}_{11}, \mathbf{I}_{22}\) are identity matrices,
and the input vector is:

\[ \mathbf{u} = \begin{bmatrix} \mathbf{b} & f_0 & f_c \end{bmatrix} \]  

(13)

The vector of temperatures \( \theta_0 \), can be obtained from the first row of the equation (9):

\[ \theta_0 = C_S \theta_c + D_S \mathbf{u} \]  

(14)

where

\[ C_S = -K_{11}^{-1} K_{12} \]  

(15)

and

\[ D_S = [K_{bb} \ I_{11} \ 0] \]  

(16)

**TRANSFER FUNCTION**

The relation between the inputs, \( \mathbf{u} \), and the output, \( \theta_a \), can be expressed as a transfer function. Applying Laplace transform for zero initial conditions, the set of equations (10) become:

\[ \theta_a = [C_S (sI - A_S)]^{-1} B_S + D_S \mathbf{u} \]  

(17)

where

\[ H = [C_S (sI - A_S)]^{-1} B_S + D_S \]  

(18)

is the transfer function the indoor air temperature \( \theta_a \).

For the vector of inputs:

\[ \mathbf{u} = \begin{bmatrix} T_{ow} & T_{ov} & \dot{Q}_a & \dot{Q}_i & \dot{Q}_{HVAC} \end{bmatrix} \]  

(19)

and for the values given in Table 1, the transfer function

\[ H = [H_1 \ H_2 \ H_3 \ H_4 \ H_5] \]  

(20)

for non-negligible indoor air heat capacity is

\[ \begin{bmatrix} H_1 \\ H_2 \\ H_3 \\ H_4 \\ H_5 \end{bmatrix} = \begin{bmatrix} 3.59 \cdot 10^{-2} \\ 1.50 \cdot 10^{-2} \cdot s^{-1} + 7.93 \cdot 10^{10} \cdot s^{-2} + 6.56 \cdot 10^{9} \cdot s^{-1} + 1 \\ 9.64 \cdot 10^{-1} \cdot (7.29 \cdot 10^{10} \cdot s^{-2} + 6.26 \cdot 10^{9} \cdot s^{-1} + 1) \\ 1.50 \cdot 10^{-14} \cdot s^{-3} + 7.93 \cdot 10^{10} \cdot s^{-2} + 6.56 \cdot 10^{9} \cdot s^{-1} + 1 \\ 1.43 \cdot 10^{-4} \cdot s^{-1} \\ 1.50 \cdot 10^{-14} \cdot s^{-3} + 7.93 \cdot 10^{10} \cdot s^{-2} + 6.56 \cdot 10^{9} \cdot s^{-1} + 1 \\ 2.49 \cdot 10^{-2} \cdot (7.13 \cdot 10^{10} \cdot s^{-2} + 6.17 \cdot 10^{9} \cdot s^{-1} + 1) \\ 1.50 \cdot 10^{-14} \cdot s^{-3} + 7.93 \cdot 10^{10} \cdot s^{-2} + 6.56 \cdot 10^{9} \cdot s^{-1} + 1 \\ 2.52 \cdot 10^{-2} \cdot (7.29 \cdot 10^{10} \cdot s^{-2} + 6.26 \cdot 10^{9} \cdot s^{-1} + 1) \\ 1.50 \cdot 10^{-14} \cdot s^{-3} + 7.93 \cdot 10^{10} \cdot s^{-2} + 6.56 \cdot 10^{9} \cdot s^{-1} + 1 \end{bmatrix} \]

(21)

By using the transfer functions, the output is

\[ \theta_a = H_1 T_{ow} + H_2 T_{ov} + H_3 \dot{Q}_a + H_4 \dot{Q}_i + H_5 (\dot{Q}_g + \dot{Q}_{HVAC}) \]

(22)

Equation (21) is a simulation problem: transfer functions are proper or strictly proper.

**TRANSFER FUNCTION OF THE THERMAL LOAD**

In heat balance method, the thermal load results from Equation (21), according to Equations (2) and (3):

\[ \dot{Q}_{HVAC} = \dot{H}_5^{-1} \dot{Q}_a - \dot{H}_5^{-1} (H_1 + H_2) T_{ow} \]

(23)

If the thermal capacity of the indoor air is non-negligible, the transfer function \( \dot{H}_5^{-1} \), corresponding to input \( \dot{Q}_a \), is improper. The transfer functions \( \dot{H}_5^{-1} H_2 \), relating \( \dot{Q}_{HVAC} \) to the outdoor temperature \( T_{ov} \) through ventilation, and \( \dot{H}_5^{-1} H_4 \) relating \( \dot{Q}_{HVAC} \) to the heat flow rate on the indoor walls are proper. They correspond to inputs connected directly to the node \( \dot{Q}_a \), i.e. not through a node containing a thermal capacity. The functions \( \dot{H}_5^{-1} H_1 \) corresponding to the influence of the outdoor temperature, \( T_{ow} \), through walls and \( \dot{H}_5^{-1} H_3 \) corresponding to influence of the heat flux on the outdoor wall, \( \dot{Q}_a \), are strictly proper. They correspond to inputs connected to the node \( \dot{Q}_a \) through nodes containing at least a thermal capacity.
The method for load calculation based on Equation (2) results in an improper function for input $a\theta$ if the heat capacity of the air is non-negligible. This is true not only for the epitome example but for any model constructed on this method. For any thermal model constructed according to the heat balance method for load estimation:

- the transfer function relating the load $\dot{Q}_{H_{VAC}}$ to the temperature in node $\theta$ will be always improper (the degree of the numerator is larger than the degree of the denominator),
- the transfer functions relating the inputs to the output $\dot{Q}_{H_{VAC}}$ through branches which do not contain at least a thermal capacity will be strictly proper and
- the transfer functions connecting the inputs to the output $\dot{Q}_{H_{VAC}}$ by branches containing at least a thermal capacity will be proper.

**CONSEQUENCES OF ACAUSALITY**

The Bode diagrams of the acausal transfer function $H^{-1}_5 = \dot{Q}_{H_{VAC}} / \theta_a$ show the problem induced by a simulation time step lower than the cut-off frequency. The magnitude of the transfer function increases with the frequency. This means that the calculated thermal load tends to infinity if the simulation time-step tends to zero. In our example, for simulation time step larger than 3h, the high frequencies will be cut and the amplification will be around 32 dB (i.e. for $u = 1$, it results $y = 10^{32/20} = 39.8$). For simulation time step smaller than 3h, the output will depend on the time step. For example, the amplification will be around 61 dB (i.e. for $u = 1$, it results $y = 10^{61/20} = 1122$) for the time step around 1 min.

In order to see the consequences of acausality in time-domain, let us simulate the response of the transfer function $H^{-1}_5 = \dot{Q}_{H_{VAC}} / \theta_a$ to a step input. The integration can be done by using any method. For the sake of simplicity, let us consider the Euler method. In this case

$$s \leftrightarrow (1 - q^{-1}) / \Delta t \quad (23)$$

where $s = \sigma + j\omega$ is the complex variable and $q^{-1}$ is the one-step delay operator. The transfer function $H^{-1}_5$ is

$$\frac{\dot{Q}_{H_{VAC}}}{\theta_a} = -22.1q^{-3} + 108.2q^{-2} - 152.3q^{-1} + 65.2$$

for the simulation time-step $\Delta t = 3600s$ and
The practical consequence of the acausal model of thermal load calculation appears when the thermal capacity of the indoor air is non-negligible and when the value of the indoor air temperature varies. The formulation of the problem requires that the load needs to change the indoor temperature in a few time steps of simulation (depending on the numerical method used, e.g., precisely one time step for Euler method). Consequently, the maximum load calculated with this method will depend on the size of the time step use in simulation.

The solution to this problem is to transform the acausal model into a causal model. This can be done by introducing a filter that achieves the regularization. In fact, the problem of finding the thermal load is a control problem and can be treated with the methods used in control systems theory.

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